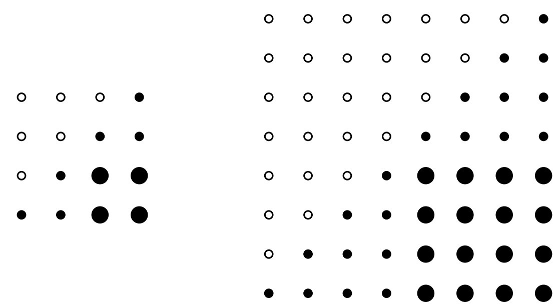
Algorithms: Some asymptotic sums

Model 1: Three proofs



$$\begin{array}{ccccccccccc}
1 & + & 2 & + & 3 & + & \dots & + & (n-1) & + & n \\
n & + & (n-1) & + & (n-2) & + & \dots & + & 2 & + & 1 \\
\hline
(n+1) & + & (n+1) & + & (n+1) & + & \dots & + & (n+1) & + & (n+1)
\end{array}$$

$$\begin{aligned}
1+2+\cdots+n &< n+n+\cdots+n && (1)\\
&= n^2 && (2)
\end{aligned}$$

$$\begin{aligned}
1+2+\cdots+n &> n/2+(n/2+1)+\cdots+n && (3)\\
&> n/2+n/2+\cdots+n/2 && (4)\\
&= (n/2)^2 && (5)\\
&= n^2/4 && (6)
\end{aligned}$$

Learning objective: Students will understand and prove the asymptotic behavior of $1 + 2 + 3 + \cdots + n$.

Learning objective: Students will apply geometric, algebraic, and inequational reasoning to asymptotic behavior.

The first row of Model 1 actually shows two similar diagrams at different sizes, one 4×4 and one 8×8. Each diagram consists of a bunch of dots—some hollow and some filled; and the filled dots come in two varieties, big and small.

1 How many dots are there in total in the first diagram? In the second diagram?

2 How many big dots are there (*i.e.* the lower-right square) in the first diagram? How many are in the second?

3 How many filled dots are there in total (both big and small filled dots, *i.e.* the lower-right triangle) in the first diagram? In the second?

Now suppose that we abstract away the specific sizes of the diagrams and imagine a generic $n \times n$ version of the same diagram. To make things slightly simpler, assume that n is even.

4 In terms of n, how many dots would there be in total?

5 In terms of n, how many big dots would there be in the lower right?

6 Explain why the number of filled-in dots is equal to

$$1 + 2 + 3 + \cdots + n.$$

7 Based on the diagrams, write an inequality relating these three quantities.

8 What does this prove about the sum $1 + 2 + 3 + \cdots + n$ in terms of Θ? Justify your answer based on your answer to the previous question.

Now consider the second proof.

9 Notice that the top row is our friend $1 + 2 + 3 + \cdots + n$. What is the second row?

10 Why does the bottom row consist of copies of $(n + 1)$?

11 What is the sum of the bottom row?

12 Use this to derive a formula for $1 + 2 + \cdots + n$ in terms of n.

13 What does this formula imply about the asymptotic behavior of $1 + 2 + \cdots + n$ in terms of Θ? Justify your answer.

Finally, consider the third proof. Surprise!—once again it has to do with the sum $1 + 2 + 3 + \cdots + n$. For this proof we will again assume n is even.[1]

[1] It is not hard to fix the proof to work for odd n as well, but the details would end up obscuring the main idea somewhat.

14 Why is step (1) true?

15 Why is the right-hand side of (1) equal to (2) ?

16 What does this prove about $1 + 2 + \cdots + n$?

17 Now, what is happening in step (3)?

18 Why is the right-hand side of (3) greater than (4)?

19 Why is (4) equal to (5)?

20 What does this prove about $1 + 2 + \cdots + n$ in terms of Θ?

21 Two of these three proofs are in some sense the same. Which two?