

Algorithms: Graphs

Model 1: Graphs and graph terms

The image displays four distinct graph structures within a rectangular frame.

1. **Numeric graph:** A connected graph with vertices labeled 0 through 9. Vertex 0 is connected to 1 and 2. Vertex 1 is connected to 0, 2, and 3. Vertex 2 is connected to 0, 1, 5, and 8. Vertex 3 is connected to 1 and 4. Vertex 4 is connected to 3 and 9. Vertex 5 is connected to 2, 6, 7, and 8. Vertex 6 is connected to 5 and 7. Vertex 7 is connected to 5 and 6. Vertex 8 is connected to 2, 5, and 7. Vertex 9 is connected to 3.

2. **Lowercase graph:** A tree-like structure with vertices a through j. Vertex a is connected to c. Vertex b is connected to c. Vertex c is connected to a, b, and d. Vertex d is connected to c, e, and g. Vertex e is connected to d and f. Vertex f is connected to e. Vertex g is connected to d, h, i, and j. Vertex h is connected to g. Vertex i is connected to g. Vertex j is connected to g.

3. **Greek graph:** A graph with vertices alpha, beta, gamma, delta, epsilon, zeta, eta, theta. Edges connect {alpha, delta}, {theta, eta}, {beta, alpha}, {zeta, delta}, {epsilon, eta}, and {gamma, alpha}.

4. **Uppercase graph:** A graph with vertices A through H. Edges connect D-E, E-F, F-A, C-B, C-A, and C-D. Vertices G and H are isolated.

$G = (V, E)$
 $V = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta\}$
 $E = \{\{\alpha, \delta\}, \{\theta, \eta\}, \{\beta, \alpha\}, \{\zeta, \delta\}, \{\epsilon, \eta\}, \{\gamma, \alpha\}\}$

Definition 1. A (simple) graph $G = (V, E)$ is a set of vertices V together with a set E of edges, where each edge consists of a set of two vertices.

Above are shown four example graphs. In clockwise order from the upper-left, we will refer to them as the numeric graph, the lowercase graph, the Greek graph, and the uppercase graph. Here are some facts about the graphs:

- C and B are *adjacent* (aka *neighbors*). So are 6 and 5. C and F are not adjacent. f and d are not adjacent. H is not adjacent to itself.
- The *degree* of vertex C is 2. The degree of vertex H is 0. The degree of vertex g is 4.
- Vertices $D, F, 4, 9, a, b, f, h, i, j$ are *leaves*. No other vertex is a leaf.

Don't worry if you don't remember all these graph terms! In fact you should probably try to forget what you think you might remember and just focus on the information in the model. Part of the point of this exercise is to help you either recall these terms, or learn them for the first time.

Learning objective: Students will understand and apply graph terms *edge, vertex, adjacent/neighbor, degree, leaf, path, connected, connected component, cycle, cyclic, acyclic, and tree.*

- 1 What does it mean for two vertices to be *adjacent*?
- 2 Are α and β adjacent? What about α and ζ ?
- 3 What is the *degree* of a vertex? Use the term *adjacent* in your answer.
- 4 What is the degree of α ?
- 5 What is the definition of a *leaf*? Use the term *degree* in your answer.
- 6 Which vertices are leaves in the Greek graph?



Model 2: More graph terms

Here are a few more facts about the graphs from Model 1:

- Vertices 7–8–5–2–0 are a *path*. 2–1–0–2–1 is also a path. H by itself is a path. $H-F-B$ is not a path. 8–2–1 is also not a path.
- Vertices f and g are *connected*. So are 7 and 8. C and F are not connected. H is connected to itself.
- The graph with numbers is a *connected graph*. So is the graph with lowercase letters. The graph with uppercase letters is not a connected graph (it is *disconnected*).
- The numeric and lowercase graphs have one *connected component* each. The uppercase graph has four connected components.
- 0–1–2–0 is a *cycle*. So is $C-A-B-C$. $h-g-i-h$ is not a cycle. $h-g-h$ is not a cycle either. Nor is $H-H$.
- The numeric graph and uppercase graph are *cyclic* graphs. The lowercase graph is *acyclic*.
- The lowercase graph is a *tree*. None of the other graphs are trees (not even the Greek one).

- 7 How many vertices can be in a *path*?
- 8 What do you think is the definition of a *path*?
- 9 Give an example of a path in the Greek graph.
- 10 Can two vertices be connected but not adjacent? If so, give an example.
- 11 Can two vertices be adjacent but not connected? If so, give an example.



- 12 What do you think it means for two vertices to be connected? Be sure to use the term *path* in your answer.
- 13 Is the Greek graph connected?
- 14 What do you think is the definition of a *connected graph*?
- 15 How many vertices can be in a connected component?
- 16 How many *connected components* does the Greek graph have?
- 17 Is the set of vertices $\{E, F\}$ a connected component? Why or why not?
- 18 Write a definition for *connected component*.
- 19 Write an “if and only if” statement using the terms *connected graph* and *connected component*.
- 20 What is a *cycle*? Use the term *path* in your answer.
- 21 Does your definition for *cycle* correctly explain why $h-g-h$ is not a cycle? If not, revise it so it does.
- 22 Does the Greek graph have a cycle?



- 23 Is the Greek graph cyclic or acyclic?
- 24 What do you think is the definition of a tree? You should use two of the other graph terms in your definition.

Warning—a tree graph is not quite the same thing as a tree data structure!



Some proofs about graphs

Theorem 2 (Trees). Let $G = (V, E)$ be a graph with $|V| = n \geq 1$. Any two of the following imply the third:

1. G is connected.
2. G is acyclic.
3. G has $n - 1$ edges.

We will take each pair of statements in turn and show that they imply the third. Fill in the blanks to complete the following proofs! Note that the size of a blank does not necessarily correspond to the amount of stuff you should write in it.

Lemma 3. (1), (2) \implies (3). That is: let $G = (V, E)$ be a graph with

$|V| = n \geq 1$. If _____

and _____,

then _____.

Proof. Let $P(n)$ denote the statement “Any graph G with n vertices

which is _____ and _____

must have _____.”

We wish to show that $P(n)$ holds for all $n \geq 1$.

The proof is by _____.

- The base case is when _____.

In this case, G must be _____

which indeed _____.

- For the induction step, suppose $P(k)$ holds for some $k \geq 1$. That is,

suppose that any graph with _____ vertices

which is _____

must have _____.



Learning objective: Students will write proofs about graphs.

Then we wish to show $P(k+1)$, that is, any graph with _____ vertices which is connected and acyclic must have _____.

So, let G be a graph with _____ vertices which is _____ and _____.
We claim that G must have some vertex which is a leaf, that is, a vertex of degree _____:

- G cannot have any vertices of degree _____ because _____.
- It also cannot be the case that every vertex of G has degree \geq _____.

If they did, then we could find a _____ by starting at any vertex and walking along edges randomly until _____; we would never get stuck because _____.

However, this is impossible because we assumed _____.

Hence, G must have some vertex which _____.
If we delete this vertex along with the edge adjacent to it, it results

in a graph G' with only _____ vertices;

we note that G' is still _____

because _____

and also _____

because _____.

Hence we may apply the inductive hypothesis to conclude that G'

_____. Adding the deleted vertex and edge



back to G' shows that G _____,
which is what we wanted to show.

□

Let's do one more! (You will do the third on your HW.)

Lemma 4. (2), (3) \implies (1), that is, _____

_____.

Proof. This proof uses a *counting argument*: we will show what we wish to show by counting things in multiple ways.

Let c denote the number of connected components of G . We want

to show that _____.

Number the components of G from $1 \dots c$, and say that component i has n_i vertices. Then

$$\sum_{i=1}^c n_i = \underline{\hspace{2cm}}$$

because _____.

Each connected component is by definition a _____ graph;

each component must also be _____

since we assumed that G is. Hence we may apply Lemma 3 to con-

clude that component i _____.

Adding these up, the total number of edges in G is

$$|E| = \sum_{i=1}^c \underline{\hspace{1cm}} = \underline{\hspace{2cm}}$$

But we already assumed the number of edges in G is _____,

and hence _____ as desired. □

