Algorithms: Some Proofs about Trees

Model 1: A theorem about trees

Learning objective: Students will write proofs about graphs.

> **Theorem 1** (Trees). *Let $G = (V, E)$ be a graph with $|V| = n \geq 1$. Any two of the following imply the third:*
>
> 1. *G is connected.*
> 2. *G is acyclic.*
> 3. *G has $n - 1$ edges.*

1 (Review) From the previous activity, what is the definition of a tree?

2 How does the given theorem relate to the definition of a tree?

3 The theorem implies that there are two other alternative, yet equivalent, ways we could have defined trees. What are they?

We will take each pair of statements in turn and show that they imply the third. Fill in the blanks to complete the following proofs! Note that the size of a blank does not necessarily correspond to the amount of stuff you should write in it.

Lemma 2. $(1), (2) \implies (3)$. *That is: let $G = (V, E)$ be a graph with*

$|V| = n \geq 1$. *If* ____

and ____,

then ____.

Proof. Let $P(n)$ denote the statement "Any graph G with n vertices

which is ________________________ and ________________________

must have ____."
We wish to show that $P(n)$ holds for all $n \geq 1$.

The proof is by ____.

- The base case is when ____.

  In this case, G must be ____

  which indeed ____.

- For the induction step, suppose $P(k)$ holds for some $k \geq 1$. That is,

  suppose that any graph with __________________________ vertices

  which is ____

  must have ____.

  Then we wish to show $P(k+1)$, that is, any graph with __________

  vertices which is connected and acyclic must have ______________.

  So, let G be a graph with _____________________ vertices which is

  ___________________________ and ___________________________.
  We claim that G must have some vertex which is a leaf, that is, a

  vertex of degree ____,
  which we can show as follows:

  – G cannot have any vertices of degree ________________________

    because ____.

  – It also cannot be the case that every vertex of G has degree $\geq$ __.

    If they did, then we could find a ____________ by starting at any vertex and walking along edges randomly until ____________;

we would never get stuck because ______________________.

However, this is impossible because we assumed ___________.

Hence, G must have some vertex which ____________________. If we delete this vertex along with the edge adjacent to it, it results in a graph G' with only __________________________ vertices; we note that G' is still _________________________________ because ____ and also ____ because ____. Hence we may apply the inductive hypothesis to conclude that G' ______________________. Adding the deleted vertex and edge back to G' shows that G ______________________________, which is what we wanted to show.

□

Let's do one more! (You will do the third on your HW.)

Lemma 3. $(2),(3) \implies (1)$, *that is,* ____.

Proof. This proof uses a *counting argument*: we will show what we wish to show by counting things in multiple ways.

Let c denote the number of connected components of G. We want to show that ____.

Number the components of G from $1 \dots c$, and say that component i has n_i vertices. Then

$$\sum_{i=1}^{c} n_i = \_\_\_\_\_\_\_\_\_\_$$

because ____.

Each connected component is by definition a ____________ graph;

each component must also be __________________________
since we assumed that G is. Hence we may apply Lemma 2 to conclude that component i ______________________________.
Adding these up, the total number of edges in G is

$$|E| = \sum_{i=1}^{c} \underline{\hspace{2cm}} = \underline{\hspace{4cm}}$$

But we already assumed the number of edges in G is __________,

and hence _______________ as desired. □