# *Algorithms: Some Proofs about Trees (Key)*

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## *Model 1: A theorem about trees*

**Theorem 1** (Trees). *Let $G = (V, E)$ be a graph with $|V| = n \geq 1$. Any two of the following imply the third:*

1. *$G$ is connected.*
2. *$G$ is acyclic.*
3. *$G$ has $n - 1$ edges.*

**Lemma 2.** $(1), (2) \implies (3)$. *That is: let $G = (V, E)$ be a graph with $|V| = n \geq 1$. If $G$ is connected and acyclic, then $G$ has $n - 1$ edges.*

*Proof.* Let $P(n)$ denote the statement "Any graph $G$ with $n$ vertices which is connected and acyclic must have $n - 1$ edges." We wish to show that $P(n)$ holds for all $n \geq 1$.

The proof is by induction on $n$.

- The base case is when $n = 1$. In this case, $G$ is just a single vertex, so it is indeed connected, acyclic, and has $0$ edges.
- For the induction step, suppose $P(k)$ holds for some $k \geq 1$. That is, suppose that any graph with $k$ vertices which is connected and acyclic must have $k - 1$ edges. Then we wish to show $P(k + 1)$, that is, any graph with $k + 1$ vertices which is connected and acyclic must have $k$ edges.

  So, let $G$ be a graph with $k + 1$ vertices which is connected and acyclic. We claim that $G$ must have some vertex which is a leaf, that is, a vertex of degree $1$, which we can show as follows:

  - $G$ cannot have any vertices of degree $0$, because it is connected (and has at least two vertices).
  - It also cannot be the case that every vertex of $G$ has degree $\geq 2$. If they did, then we could find a cycle by starting at any vertex and walking along edges randomly until encountering a repeated vertex; we would never get stuck because every vertex has degree $\geq 2$, that is, if we come in along one edge there must always be a different edge along which we can leave. However, this is impossible because we assumed $G$ is acyclic.

Hence, $G$ must have some vertex which is a leaf. If we delete this vertex along with the edge adjacent to it, it results in a graph $G'$ with only $k$ vertices; we note that $G'$ is still connected (because $G$ was connected and we deleted a leaf) and also acyclic (because deleting something from an acyclic graph cannot create a cycle). Hence we may apply the inductive hypothesis to conclude that $G'$ has $k-1$ edges. Adding the deleted vertex and edge back to $G'$ shows that $G$ has $k$ edges, which is what we wanted to show.

□

**Lemma 3.** $(2),(3) \implies (1)$, *that is, any acyclic graph with $n$ vertices and $n-1$ edges must be connected.*

*Proof.* This proof uses a *counting argument*: we will show what we wish to show by counting things in multiple ways.

Let $c$ denote the number of connected components of $G$. We want to show that $c = 1$.

Number the components of $G$ from $1 \dots c$, and say that component $i$ has $n_i$ vertices. Then

$$\sum_{i=1}^{c} n_i = n$$

because adding up the number of vertices in each component gives the total number of vertices. Each connected component is by definition a connected graph; each component must also be acyclic since we assumed that $G$ is acyclic. Hence we may apply Lemma 2 to conclude that component $i$ has $n_i - 1$ edges. Adding these up, the total number of edges in $G$ is

$$|E| = \sum_{i=1}^{c}(n_i - 1) = \left(\sum_{i=1}^{c} n_i\right) - \left(\sum_{i=1}^{c} 1\right) = n - c.$$

But we already assumed the number of edges in $G$ is $n-1$, and hence $c = 1$ as desired. □