

Algorithms: Introduction to Dynamic Programming

Model 1: Fibonacci

Learning objective: Students will apply memoization techniques to speed up overlapping recursion.

Here are three functions to compute Fibonacci numbers, implemented in Python. You may assume that they are all correct.

```
def fib1(n):
    if n <= 1:
        return n
    else:
        return fib1(n-1) + fib1(n-2)

def fib2(n):
    fibs = [0] * (n+1) # Create initial array of all 0s
    fibs[1] = 1

    for i in range(2, n+1):
        fibs[i] = fibs[i-1] + fibs[i-2]

    return fibs[n]

fibtable = [0,1]

def fib3(n):

    while len(fibtable) < n+1:
        fibtable.append(-1)

    if fibtable[n] == -1:
        fibtable[n] = fib3(n-1) + fib3(n-2)

    return fibtable[n]
```

- 1 Recall that the Fibonacci numbers are defined by the recurrence

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

Which of the three implementations corresponds most directly to this definition?

- 2 Draw the call tree for `fib1(5)`.

- 3 How many times does `fib1(2)` occur in the call tree? What about `fib1(1)`? `fib1(0)`?

- 4 It turns out that `fib1` is extremely slow.¹ Intuitively, why is it so slow?

¹ In fact, it takes $\Theta(\varphi^n)$ time.

- 5 Trace the execution of `fib2(5)` and explain how it works.

- 6 Which does more work, `fib2(5)` or `fib1(5)`? Why?



7 In terms of Θ , how long does `fib2(n)` take?²

² For the purposes of this activity, you should assume that each addition takes constant time. However, as you know from a previous activity, it is more accurate to say that addition takes linear time in the number of bits, which actually makes a difference here since Fibonacci numbers can get quite large. You will analyze the situation more precisely on the HW.

8 Suppose we switch the direction of the `for` loop in `fib2`, so `i` loops from `n` down to 2. Would it still work? Why or why not?

9 Trace the execution of `fib3(5)` and explain how it works.

10 In terms of Θ , how long does `fib3(n)` take?

11 Fill in this statement: `fib3` is just like `fib1` except that

_____.

12 Fill in this statement: `fib2` is just like `fib3` except that

_____.

13 Why don't we do something akin to `fib2` or `fib3` for merge sort?

14 Consider the following recursive definition of $Q(n)$ for $n \geq 0$:

$$\begin{aligned}
 Q(0) &= 0 \\
 Q(1) &= Q(2) = 1 \\
 Q(n) &= \max \begin{cases} Q(n-3)^2 \\ Q(n-1) + Q(n-2) \end{cases}
 \end{aligned}$$



(Note that there are *three* base cases.) Using pseudocode, write an algorithm to calculate $Q(n)$ efficiently.

