

## Algorithms: Divide and conquer arithmetic

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### Model 1: Arithmetic

$$\begin{array}{r} 1011010 \\ + 1110011 \\ \hline 11001101 \end{array}$$
  
$$\begin{array}{r} 1011010 \\ \times 1110011 \\ \hline 1011010 \\ 10110100 \\ 101101000 \\ + 1011010000 \\ \hline 10100001110 \end{array}$$

In many situations, when we use arithmetic operations on integers such as addition or multiplication, we simply assume they take constant time. This is appropriate when the integers are bounded by some fixed size, *e.g.* if all the integers are 64-bit integers. However, if we want to be able to deal with integers of arbitrary size, this is no longer appropriate; we must take the size of the integers into account.

**Learning objective:** Students will analyze arithmetic operations on  $n$ -bit integers.

- 1 Approximately how many single bit operations (*e.g.* adding or comparing two bits) are needed to compute the addition  $1011010_2 + 1110011_2$  shown in the model?
- 2 In general, suppose we want to add two integers of  $n$  bits each. In terms of  $\Theta$ , how many bit operations are needed?

- 3 Explain why it is not possible to do any better than this.
- 4 If we have a list of  $\Theta(n)$  integers, each with  $\Theta(n)$  bits, how long will it take (in terms of single bit operations) to add all of them?

Now consider the multiplication shown in the model.

- 5 Why are there five rows in between the two horizontal lines?
- 6 How many operations are needed to produce each such row?  
(*Hint*: you may assume that multiplying by a power of two takes constant time.)
- 7 If we are multiplying two integers with  $n$  bits each, how many intermediate rows could there be in the worst case?
- 8 How long will it take to add them all?



*Model 2: Arithmetic by pieces*

$$X = 01101001_2$$

$$Y = 11100100_2$$

$$X_1 = 0110_2$$

$$Y_1 = 1110_2$$

$$X_2 = 1001_2$$

$$Y_2 = 0100_2$$

Let's now consider whether it is possible to multiply two  $n$ -bit integers any faster.

- 9 What is the relationship between  $X$ ,  $X_1$  and  $X_2$  in the model? What about  $Y$ ,  $Y_1$ , and  $Y_2$ ?
- 10 Suppose  $Z = 1011100101_2$ . What would  $Z_1$  and  $Z_2$  be?
- 11 What is  $2^4 \cdot X_1$  in binary?
- 12 In general, if  $b$  is some number expressed in binary, what is  $2^4 \cdot b$ ?
- 13 Write two equations expressing  $X$  in terms of  $X_1$  and  $X_2$  and  $Y$  in terms of  $Y_1$  and  $Y_2$ .
- 14 In general, suppose  $A$  is an  $n$ -bit integer, and we split it into  $A_1$  and  $A_2$ . Generalize your previous answer to express  $A$  in terms of  $A_1$  and  $A_2$ .



- 15 Suppose  $A$  and  $B$  are  $n$ -bit integers, and consider the product  $AB$ . Expand both  $A$  and  $B$  using your previous answer, and distribute the resulting product. You should end up with an expression involving only  $A_1, A_2, B_1,$  and  $B_2$ .
- 16 How many multiplications are required to compute the expression from Question 15? (Remember that multiplying by a power of two takes constant time and does not need to be counted.)
- 17 How big (how many bits) are the inputs to each multiplication in Question 15?
- 18 Explain how we can use the equation from Question 15 as a recursive algorithm to compute  $AB$ .
- 19 Let  $M(n)$  denote the time taken to multiply two  $n$ -bit integers, and write a recurrence relation for  $M(n)$  corresponding to this recursive algorithm.

