Algorithms: Asymptotic Analysis: definitions

Model 1: Definitions

Definition 1 (Big-O). $T(n)$ is $O(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) \leq c \cdot g(n).$$

Definition 2 (Big-Omega). $T(n)$ is $\Omega(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) \geq c \cdot g(n).$$

Definition 3 (Big-Theta). $T(n)$ is $\Theta(g(n))$ if it is both $O(g(n))$ and $\Omega(g(n))$.

Sample proof that $n^2 + 2n$ is $\Theta(n^2)$:

- First, $n^2 + 2n \leq n^2 + 2n^2 = 3n^2$ for $n \geq 1$ (since $n^2 \geq n$ when $n \geq 1$). Hence $n^2 + 2n$ is $O(n^2)$ according to the definition if we pick $c = 3$ and $n_0 = 1$.
- Next, $n^2 + 2n \geq n^2$ as long as $n \geq 0$. So by picking $c = 1$ and $n_0 = 0$, we see that $n^2 + 2n$ is also $\Omega(n^2)$.

1 Compare our class consensus definition of $O(n^2)$ from the previous activity with the formal definition of $O(g(n))$ above. List one way in which they are similar, and one way in which they are different.

2 Consider the following three more intuitive phrasings. Match each one with its corresponding definition.

- $T(n)$ is eventually bounded below by some constant multiple of $g(n)$.
- $T(n)$ is eventually bounded between two constant multiples of $g(n)$.
- $T(n)$ is eventually bounded above by some constant multiple of $g(n)$.

3 Which part of the definitions corresponds to the word "eventually" in Question 2?

4 In the sample proof that $n^2 + 2n$ is $O(n^2)$, the given values of c and n_0 are not the only values that would work. Given an alternate proof that $n^2 + 2n$ is $O(n^2)$ using different values of c and n_0.

5 Prove that $f(n) = 20n - 1$ is $O(n^2)$ by applying the formal definition.

6 Prove that $f(n) = n^3/10$ is $\Omega(n^2)$ by applying the formal definition.

7 Prove that $f(n) = 3n^2 - n + 1$ is $\Theta(n^2)$ by applying the formal definition.

8 Consider this definition[1]: $T(n)$ is $\aleph(g(n))$ if there exist a real number $c > 0$ and an integer $n_0 \geq 0$ such that for all $n \geq n_0$,

$$T(n) = c \cdot g(n).$$

(a) True or false: if $T(n)$ is $\aleph(g(n))$, then $T(n)$ is $\Theta(g(n))$. Justify your answer.

(b) True or false: if $T(n)$ is $\Theta(g(n))$, then $T(n)$ is $\aleph(g(n))$. Justify your answer.

[1] Unlike the definitions of O, Ω, and Θ, this definition is not standard; I just made it up.