

n-bit multiplication

$$A = A_1 2^{n/2} + A_2$$

$$B = B_1 2^{n/2} + B_2$$

$$AB = A_1 B_1 2^n + (A_1 B_2 + A_2 B_1) 2^{n/2} + A_2 B_2$$

bit shifts

2n-bit additions — $\Theta(n)$ time.

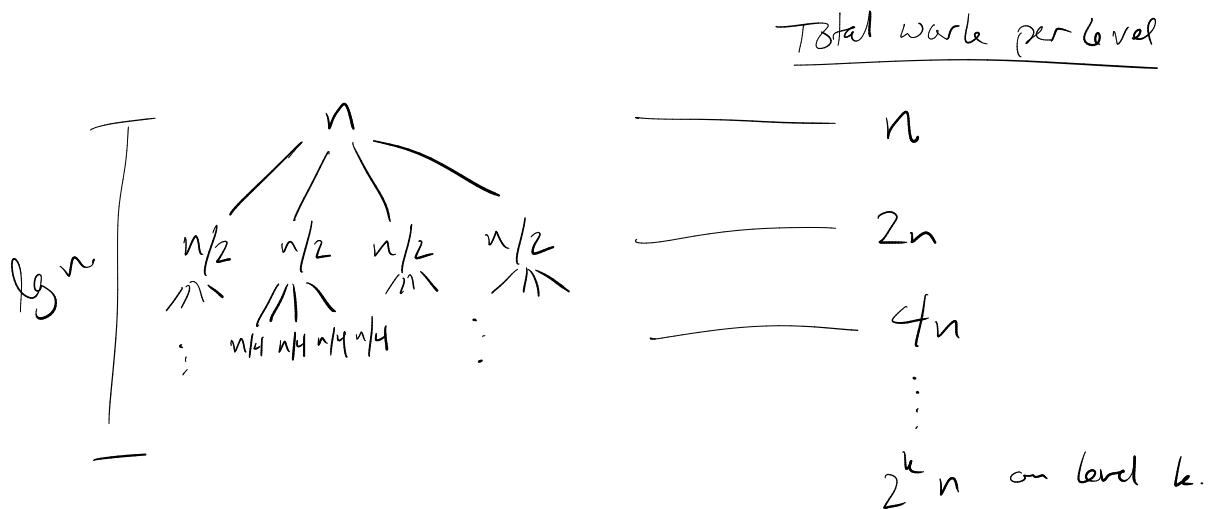
n/2-bit multiplications

This can be read as a recursive algorithm for multiplying.

time to multiply n-bit numbers

$$M(1) = \Theta(1)$$

$$M(n) = 4M(n/2) + \Theta(n).$$



So, total is:

$$n + 2n + 4n + \dots + 2^{\lg n} n$$

$$= n(1 + 2 + 4 + 8 + \dots + n)$$

$$= n(2n-1) = \Theta(n^2).$$

Not faster than usual algorithm.

$$1 + 2 + 4 + 8 + \dots + 2^k$$

$$= 2^{k+1} - 1$$

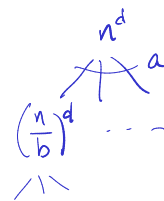
Master Theorem

If

$$\left. \begin{aligned} T(n) &= aT(n/b) + \Theta(n^d) \\ T(1) &= \Theta(1) \end{aligned} \right\}$$

then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$



$$\begin{aligned} n^d \\ a(n/b)^d \\ a^2(n/b^2)^d \\ \vdots \\ (a/b^d)^k n^d \end{aligned}$$



work decreases



work stay same



work increases

Examples

- Merge sort. $a=2$ $b=2$ $d=1$

$$a = b^d \quad 2 = 2^1 \checkmark \rightarrow \text{Case 2 of Master Thm.}$$

$$\text{So it takes } \Theta(n^d \lg n) = \Theta(n \lg n) \checkmark$$

- Binary search. $a=1$ $b=2$ $d=0$

$$\text{So } a ? b^d \rightarrow 1 = 2^0 \rightarrow \text{Case 2.}$$

$$\text{Hence } \Theta(n^d \lg n) = \Theta(\lg n).$$

- Multiplication. $a=4$ $b=2$ $d=1$

$$a ? b^d \quad 4 > 2^1 \rightarrow \text{Case 3 of Master Thm.}$$

$$\text{Hence takes } \Theta(n^{\log_b a}) = \Theta(n^{\log_2 4}) = \Theta(n^2).$$

$$A = A_1 2^{n/2} + A_2$$

$$B = B_1 2^{n/2} + B_2$$

$$P_1 = A_1 B_1$$

$$P_2 = A_2 B_2$$

$$P_3 = (A_1 + A_2)(B_1 + B_2)$$

$$= A_1 B_1 + A_1 B_2 + A_2 B_1 + A_2 B_2.$$

6 2n-bit additions.

3 n/2-bit mults.

$$\text{So } AB = P_1 2^n + (P_3 - P_1 - P_2) 2^{n/2} + P_2$$

By Master Theorem,

$$a = \underline{\underline{3}}, \quad b = 2, \quad d = 1$$

$$a > b^d, \text{ so Case 3}$$

$$\text{Hence overall } \Theta(n^{\log_b a}) = \Theta(n^{\log 3}) \approx \Theta(n^{1.59...})$$