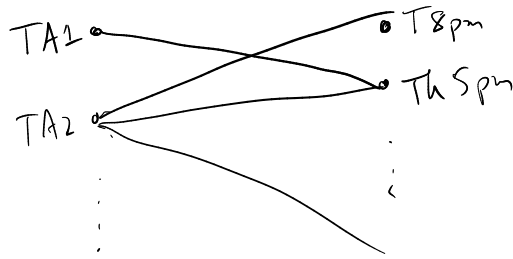


Bipartite graphs

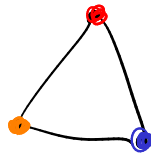
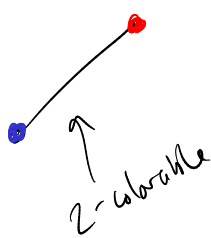
Example w/ TAs + time slots:



Vertex per TA
Vertex per time slot
Connect TAs w/ time slots
they are available.

Def'n An undirected graph $G = (V, E)$ is bipartite iff V can be partitioned into two sets L, R such that every edge has one endpoint in L and one in R .

Def'n An undirected graph is k-colorable if it is possible to assign each vertex one of k colors in such a way that no two vertices of the same color are adjacent.



← 3-colorable
but not 2-colorable.

2-colorable = bipartite

✓	✓	✓	✗
✗		✗	✓

Conjecture: odd cycle $\leftrightarrow \neg$ bipartite

(aka bipartite $\leftrightarrow$ no odd cycles)

Theorem. An undirected graph G is bipartite iff it has no odd-length cycles.

Proof we must prove both directions.

($\rightarrow$) ie. G bipartite $\rightarrow$ no odd cycles.

If G is bipartite, any path must alternate between L and R . Hence any path that returns to where it started must have even length.

($\leftarrow$) ie. no odd cycles $\rightarrow G$ bipartite.

Suppose G has no odd cycles. We must show it is bipartite, ie. we must demonstrate an appropriate partition L, R .

Run BFS starting at an arbitrary vertex. Then define

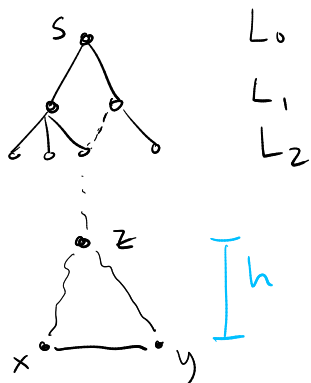
$L = L_0 \cup L_2 \cup L_4 \cup \dots$ $\leftarrow$ all even layers

$R = L_1 \cup L_3 \cup L_5 \cup \dots$ $\leftarrow$ all odd layers.

Every edge has endpoints in adjacent layers (which is ok — one end will be in L + one in R) ok in the same layer — we will show this cannot happen.

Suppose (for purposes of contradiction) there is an edge $\{x, y\}$ with both endpoints in same layer.

Consider the BFS tree:



Let z be the common ancestor of $x + y$. Suppose z is h layers above $x + y$.

$z - x - y - z$ makes a cycle of length $2h + 1$ — which is odd.

But we assumed G has no odd cycles, so there cannot actually be an edge between two vertices in the same layer.
Hence G is bipartite. 

BFS can be used to check whether a graph is bipartite —
returns L, R if so, or an odd cycle if not.