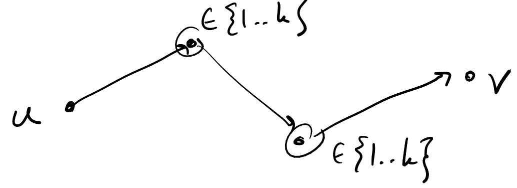
Floyd–Warshall

- Solves all-pairs shortest paths in weighted, directed graph
- Can deal w/ negative edges!

Want as output $|V| \times |V|$ matrix S such that $S[u,v]$ = shortest distance $u \to v$.

Add a parameter: Let $S[u,v,k]$ = shortest distance $u \to v$ using only vertices $1..k$ as intermediate nodes. (Assume vertices are numbered $1..n$.)

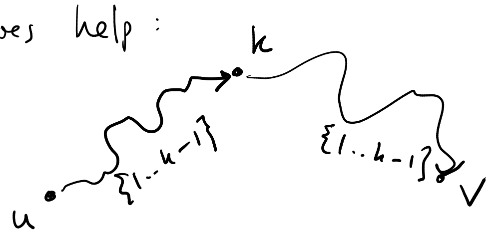


Base cases?

$$S[u,v,0] = \begin{cases} 0 & \text{if } u = v \\ w(u,v) & \text{if there is an edge } u \to v \\ \infty & \text{otherwise.} \end{cases}$$

Now suppose we already know $S[u,v,k-1]$ for all u,v. Let's see how to compute $S[u,v,k]$.

- Maybe vertex k does not help; in that case $S[u,v,k] = S[u,v,k-1]$.
- Maybe it does help:



$$S[u,v,k] = S[u,k,k-1] + S[k,v,k-1].$$

i.e. best distance $u \to v$ using $\{1..k\}$ is best distance $u \to k$ using $\{1..k-1\}$ + best distance $k \to v$ using $\{1..k-1\}$.

Overall, $$S[u,v,k] = \min \begin{cases} S[u,v,k-1] & \text{// not using } k. \\ S[u,k,k-1] + S[k,v,k-1]. & \text{// using } k \end{cases}$$

To reconstruct actual best path, we can store
$next[u,v]$ = next vertex along best path $u \to v$ so far.

We can also get away with storing only an $n \times n$ matrix S, and keep overwriting it for each k.

Floyd-Warshall(G):

$$S[u,v] \leftarrow \begin{cases} 0 & \text{if } u=v \\ w(u,v) & \text{if } u\to v \in G \\ \infty & \text{o'wise} \end{cases}$$

$O(V^3)$

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next[u,u] ← u  for all u ∈ V.
for k in 1..n:
    for u in 1..n:
        for v in 1..n:
            if S[u,k] < ∞ and S[k,v] < ∞ and
                 S[u,k] + S[k,v] < S[u,v] :
                S[u,v] ← S[u,k] + S[k,v]
                next[u,v] ← next[u,k]
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- If no negative cycles, this works fine, even w/ negative weights.
- Any vertex on a negative cycle will end up with $S[u,u] < 0$.