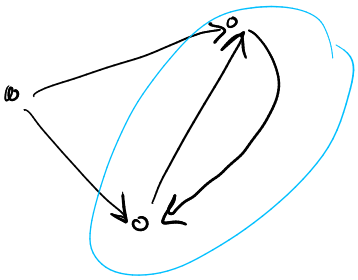
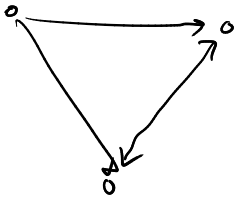
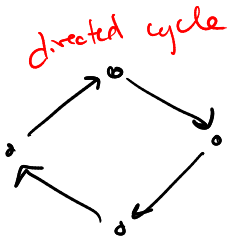
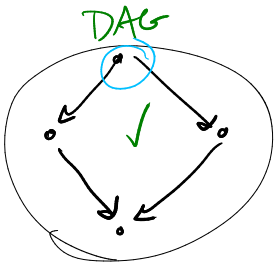
Directed graphs



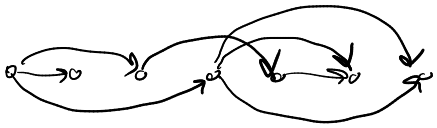


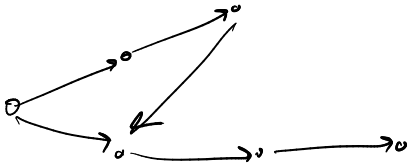
Defn A directed acyclic graph (DAG) is a directed graph with no directed cycles.





Defn A topological ordering (topological order, topological sort, topsort) of a directed graph is an ordering of the vertices such that "every edge points to the right". Formally, it is an ordering of vertices $v_1, v_2, \ldots, v_n$ such that for every edge $v_i \to v_j$, $i < j$.





Theorem. A directed graph G has a topological ordering iff G is a DAG.

Proof ($\Rightarrow$) If there is a topological ordering, there can't be any directed cycles, because any directed cycle must have at least one edge going backwards.

($\Leftarrow$) Any DAG has a top. ordering. Proof by algorithm.

Lemma Every finite DAG has a vertex with indegree 0. (# edges pointing into vertex.)

Proof Pick a vertex + follow edges backwards until getting stuck @ a vertex w/ indegree 0. You can't keep going forever since you would eventually see a vertex you'd seen before, which would make a directed cycle.

Proof (Kahn's algorithm)

- Find a vertex w/ indegree 0. Put it first in the top. order.
- Remove it (+ all its outgoing edges). The result is still a DAG, so repeat the process.