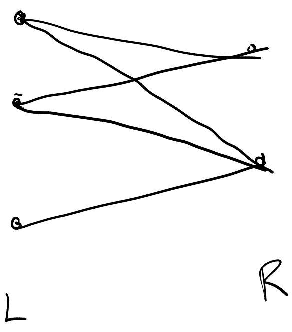
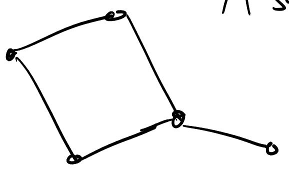
Bipartite Graphs

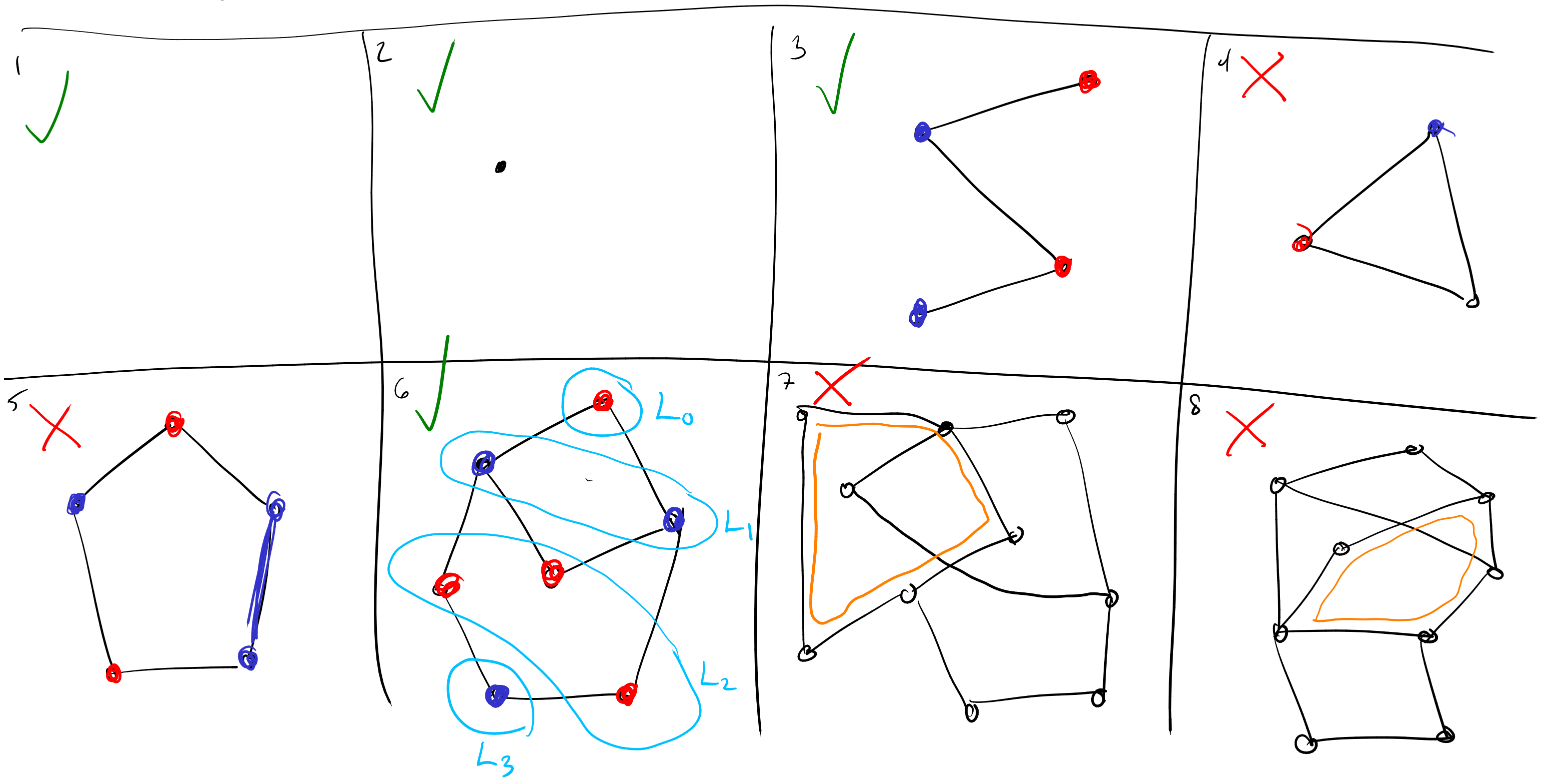


Def'n A graph $G=(V,E)$ is bipartite iff the vertices V can be partitioned into two sets L, R (red, blue, etc.) such that every edge has one end in L and one in R.

Also bipartite = "2-colorable".

same graph (not obvious!)





O = has odd cycle

B = is bipartite

$O \to \neg B$ ✓

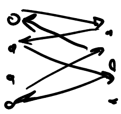
$\neg O \to B$?

Theorem. A graph G is bipartite iff it has no odd cycles. ($B \leftrightarrow \neg O$)

Proof. ($\to$) ie $B \to \neg O \equiv O \to \neg B$, if odd cycle $\to$ not bipartite.

Any odd cycle can't be assigned 2 colors; we run into problems.

(Or: any bipartite graph has only even cycles.



($\Leftarrow$) ie. $\neg O \to B$: no odd cycles $\to$ bipartite.

Proof by algorithm. Suppose we have a graph G with no odd cycles.
Pick an arbitrary vertex s and run a BFS starting from s.

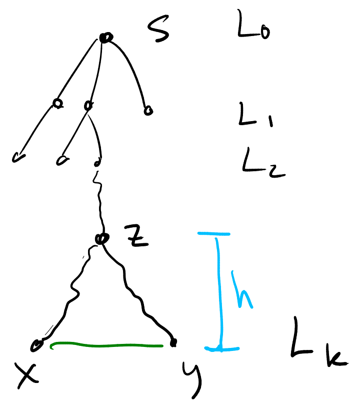
Define $$L = L_0 \cup L_2 \cup L_4 \cup \ldots$$
$$R = L_1 \cup L_3 \cup L_5 \cup \ldots$$

We must show every edge has one endpoint in L and one in R.
Recall every edge has endpoints in adjacent layers (Ok!)
or the same layer (bad!).

show this can't happen.

Suppose we did have an edge (x, y) with endpoints in the same layer.

Consider the BFS tree:



Call z the common ancestor of x + y.

There is a cycle
$z - x - y - z$
of length $2h+1$,
which is odd.

But we assumed G has no odd cycles, so (x,y) can't exist.