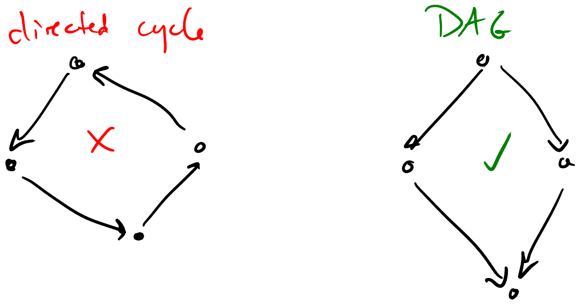
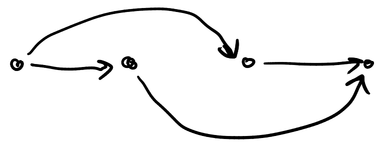
Directed graphs + topological sort

Def'n A directed acyclic graph (DAG) is a directed graph with no directed cycles.

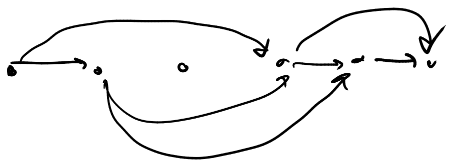


Def'n A topological ordering (topological order, topological sort, topsort) of a directed graph is an ordering of its vertices such that all edges "point to the right". More formally, it is a list of the vertices $v_1, v_2, \ldots, v_n$ such that for every edge $(v_i \to v_j)$, $i < j$.

e.g.



or e.g.



Thm A directed graph G has a topological ordering iff G is a DAG.

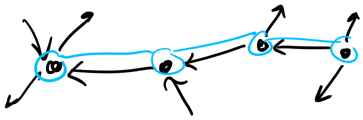
Proof. ($\Rightarrow$) If G can be drawn with every edge pointing to the right, then G cannot have a directed cycle — a directed cycle would need at least one edge pointing back to the left.

($\Leftarrow$) Proof by algorithm — we will give an algorithm to find a topological ordering for any DAG.

indegree = # of edges pointing into a vertex.

Lemma A DAG has a vertex with indegree 0.

Proof Pick any vertex and start following edges backwards until reaching a vertex with no incoming edges.



We can't keep going

forever — the graph is finite, so eventually we would reach a vertex we've seen before — but that can't happen since the graph has no directed cycles.

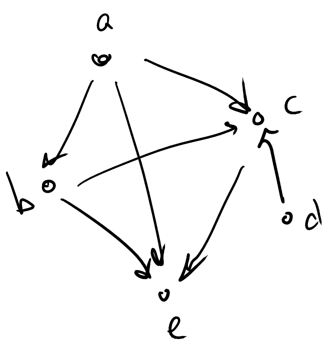
Proof that any DAG w/ n vertices has a topological ordering: proof by induction on n / recursive algorithm.

Base case: if $n = 1$ the graph is a single node, which has a top. ordering.

Inductive case: Suppose we can find a top. ordering for any DAG w/ $n-1$ vertices (IH). Given a DAG w/ n vertices, find a vertex v w/ indegree 0, put it first in the top. ordering. Delete it (+ all its outgoing edges) from G, producing another DAG G' with only $n-1$ vertices. By the IH, G' has a top. ordering which can go after v.

∎

eg.



d, a, b, c, e