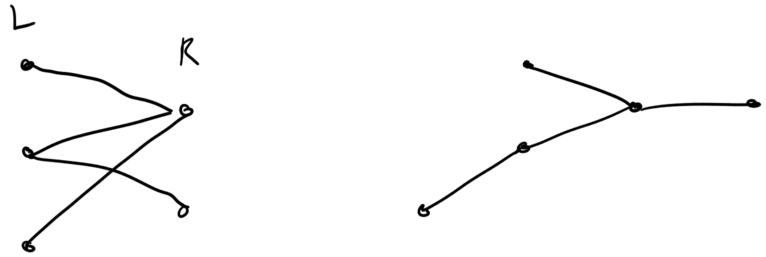
Bipartite Graphs

<u>Def'n</u> An undirected graph $G = (V, E)$ is <u>bipartite</u> if V can be partitioned into two sets L, R such that every edge has one endpoint in L and the other in R.



Bipartite also = "2-colorable": each vertex has a color, no edge may connect vertices of the same color.

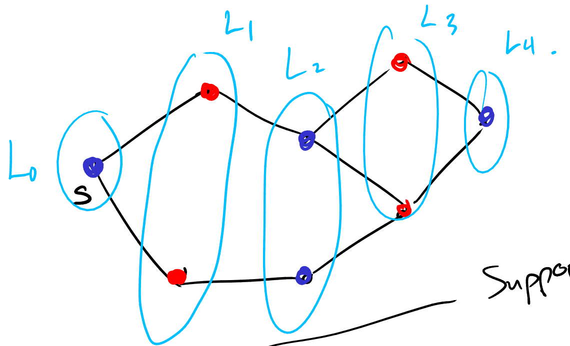
<u>Thm</u>. G is bipartite iff it has no odd-length cycles.

<u>Proof</u>. $(\Rightarrow)$ ie. if G is bipartite then it has no odd-length cycles.

Contrapositive is "if G has odd-length cycles, then it is not bipartite."

↳ Impossible to assign colors to odd-length cycle that alternate.

$(\Leftarrow)$ ie. if there are no odd-length cycles then G is bipartite.



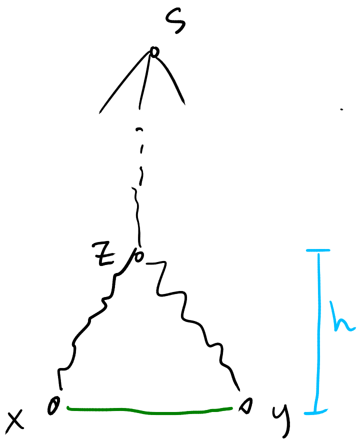
Proof by algorithm. Suppose there are no odd-length cycles. Pick some starting vertex s and run a BFS from s, generating layers $L_0, L_1, L_2, \ldots$

Then define $L = L_0 \cup L_2 \cup L_4 \cup \ldots$

$R = L_1 \cup L_3 \cup L_5 \cup \ldots$

Claim: every edge has one endpt in L and one in R.

Note every edge's endpts are either in adjacent layers (which would be fine) or the same layer (which would be a problem). Lets show this can't happen. Suppose there is some edge (x,y) where x, y are in the same layer.



L_k

Let z be the common ancestor of x, y in the BFS tree, and say the distance from z to x and y is h.

There is a cycle $z - x - y - z$ of length $2h+1$, which is odd.

But we supposed that G has no odd-length cycles, so this impossible.

ie. there cannot be an edge between vertices in the same layer. Therefore G is bipartite.