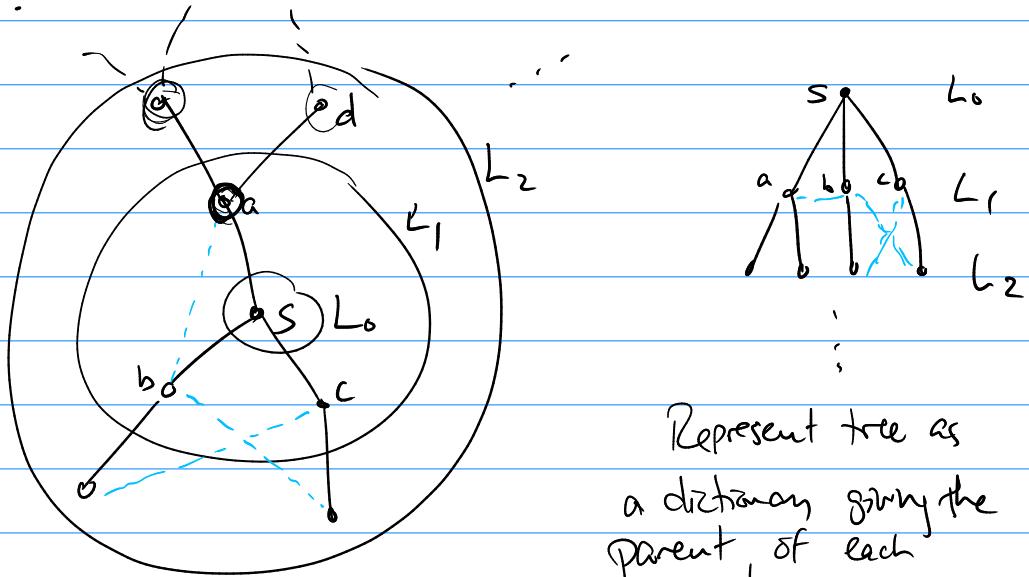


Breadth-first search.

Given an undirected, unweighted graph : answers questions:

1. Given vertices s, t in G , is there a path from s to t ? (ie. are s and t connected?)
2. Is G connected?
3. What is a shortest path from s to t ?

Idea: start @ some vertex s and explore outward in "layers".



Represent tree as
a dictionary giving the
parent of each
vertex.

Properties of BFS:

1. The shortest path from s to v has length i iff $v \in L_i$.
2. There exists a path from s to t iff t shows up in some layer of the BFS from s .
3. G is connected iff every vertex shows up in a BFS from an arbitrary starting node.
4. For each edge $(u, v) \in E$, the layers of $u + v$ differ by at most 1.

Representing graphs

1. Adjacency list representation :

Map <Vertex, List <Vertex>>

hash table

Graph start vertex

BFS (G, s) :

$Q \leftarrow$ Empty queue

parent \leftarrow empty dictionary

level \leftarrow empty dictionary

visited \leftarrow empty set

Add s to Q

Add s to visited

level [s] $\leftarrow 0$.

parent [s] $\leftarrow s$.

while Q is not empty:

remove u from Q

for each edge (u, v) in E : (for each v adjacent to u)

if $v \notin$ visited :

Add v to visited

level [v] \leftarrow level [u] + 1

parent [v] $\leftarrow u$

Add v to Q .

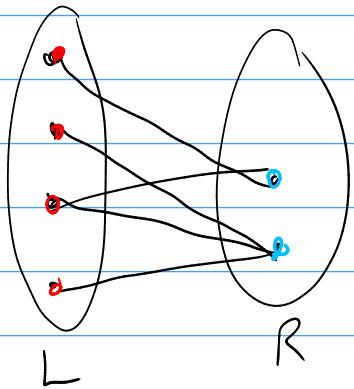
return parent, level.

TOTAL # of times this loop will execute is $2 \cdot |E|$

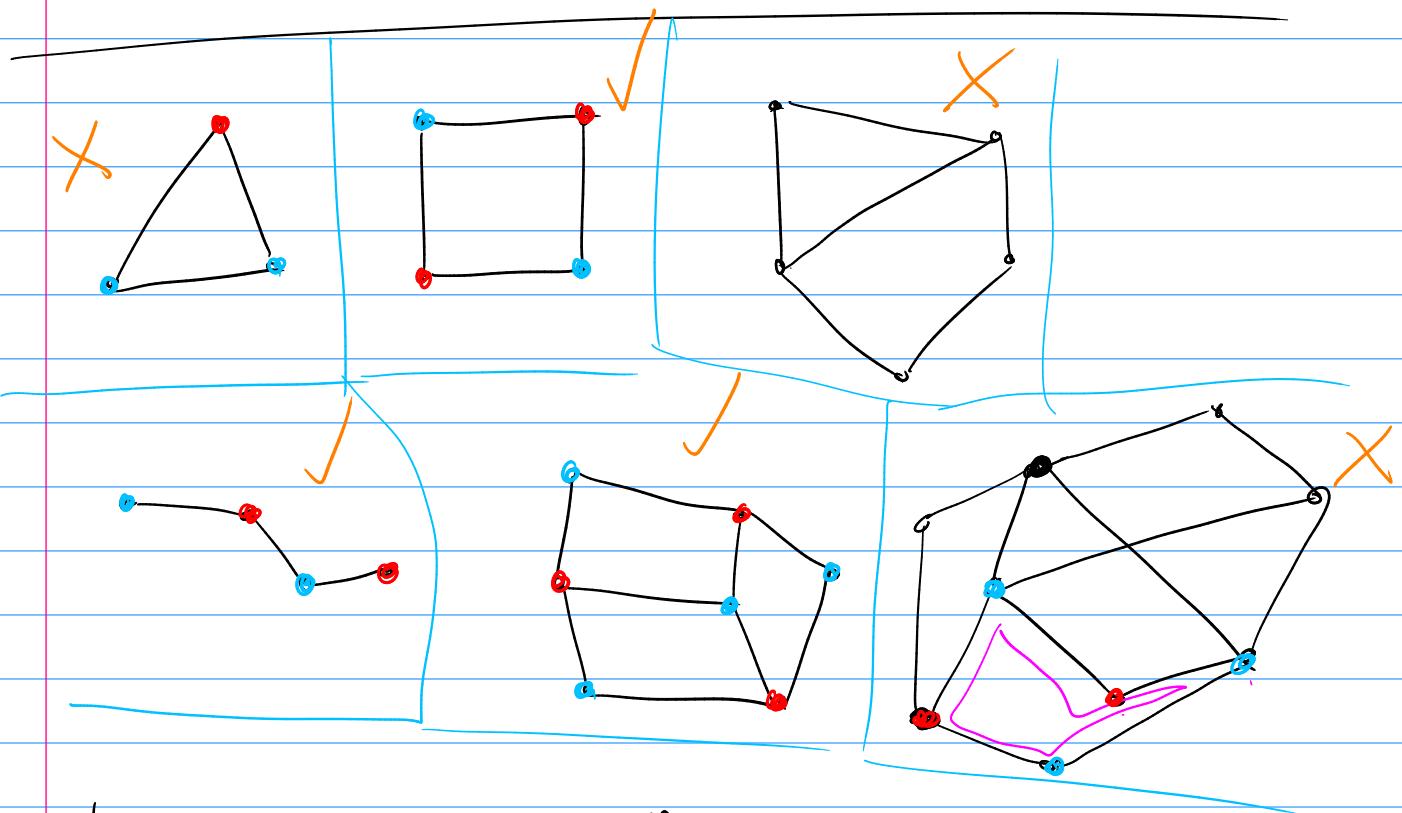
$\Theta(1)$ assuming we are using hash-based sets + dictionaries.

Hence entire algorithm is $\Theta(E)$.

Def'n An undirected graph $G = (V, E)$ is bipartite if V can be partitioned into two sets L, R such that every edge has one endpoint in L and one in R .



Can Also think in terms of red/blue



Theorem. G is bipartite iff it contains no odd-length cycles.

Proof.

(\Rightarrow) If G is bipartite, all paths must alternate between L/R (blue/red), so any cycle must have even length.

Suppose G has no odd-length cycles.

(\Leftarrow) Proof by algorithm. Pick an arbitrary starting vertex s and run BFS, generating layers L_0, L_1, L_2, \dots

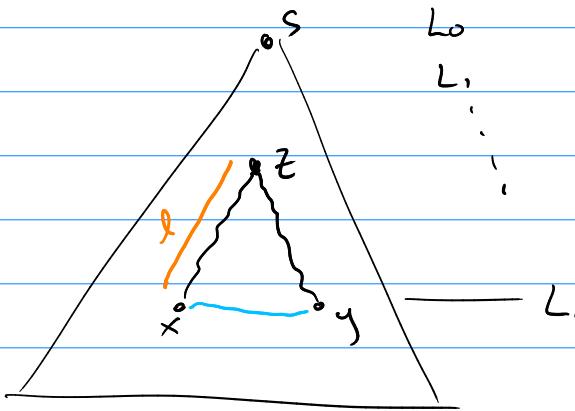
Now set red vertices = $L_0 \cup L_2 \cup L_4 \cup \dots$

blue vertices = $L_1 \cup L_3 \cup L_5 \cup \dots$

Claim: every edge has one red and one blue endpoint.

Recall every edge must be either between 2 vertices in the same layer or adjacent layers. We want to show that having 2 endpoints in same layer is not possible.

Suppose there is an edge (x, y) such that $x, y \in L_i$.



Let z be the lowest common ancestor of x, y in the BFS tree. Suppose z is distance l away from x and y .

But $z - x - y - z$ is a cycle of length $2l + 1$ which is odd, but we assumed G has no odd-length cycles. Hence there can never be an edge between 2 vertices in the same layer, so G is bipartite.



Consider generalizing BFS to directed graphs.

Def'n A directed graph is strongly connected if between any pair of vertices u, v there always exists a directed path from u to v .

Q: Given a directed graph, how can we tell whether it is strongly connected?