Name: ____________________________

CSCI 380: Activity 2

Mathematical Proofs[1]

Team Roles	**Team Member**
Facilitator: reads the questions aloud, keeps track of time and makes sure everyone contributes appropriately.	
Spokesperson: talks to the instructor and other teams.	
Quality Control: records all answers & questions, and provides team reflection to team & instructor.	
Process Analyst: Considers how the team could work and learn more effectively.	

> **Note**
>
> If you have 3 people, combine Facilitator & Process Analyst.

Learning Objectives

1. Be able to complete *proof by cases.*
2. Be able to identify the base case for a proposition to be proved by induction.
3. Be able to complete *proof by induction.*
4. Be able to complete *proof by contradiction.*
5. As a group work on improving your *Management* skill by completing the models within the time allotted.

The second critical piece of knowledge from Discrete Mathematics that you will apply in this course is mathematical proofs. This activity will review the three main proof techniques: Proof by Cases, Proof by Induction, and Proof by Contradiction.

In this Activity, you will find three Modules and one Reflection section.

[1] content derived from CS-POGIL Discrete Structures exercises by Jim Van Horn. https://cspogil.org/

start time:

Model I. Proof By Cases (10 min)

Proposition 1

For any integer, n^2 is divisible by 3 OR $n^2 \bmod 3 = 1$ (That is n^2 can be expressed as $3q + 1$ for some integer q)

Proof. There is no restriction on the value of n, so when divided by 3 it can have a remainder of: 0, 1, or 2

CASE 1: n may have a remainder of 0. In that case it is divisible by 3 and can be represented as $n = 3K$ for some integer K.

$$\begin{aligned} n^2 &= (3K)^2 && \text{by substitution} \\ &= (3K) \cdot (3K) && \text{by definition of exponentiation} \\ &= (K \cdot 3) \cdot (3K) && \text{by commutative property} \\ &= K \cdot (3 \cdot (3K)) && \text{by associative property} \\ &= (3 \cdot (3K)) \cdot K && \text{by commutative property} \\ &= 3 \cdot (3K) \cdot K) && \text{by associative property} \end{aligned}$$

but there is an integer q such that $q = (3K) \cdot K)$ by the closure property of integers for multiplication, so $n^2 = 3q$ and is therefore divisible by 3 by definition of divisibility.

CASE 2: n may have a remainder of 1. (**This will be an exercise for the team to complete**)

CASE 3: n may have a remainder of 2 when divided by 3, that is $n = 3K + 2$

$$\begin{aligned} n^2 &= (3K + 2)^2 && \text{by substitution} \\ &= 9K^2 + 12K + 4 && \text{by algebra} \\ &= 9K^2 + 12K + (3 + 1) && \text{arithmetic} \\ &= (9K^2 + 12K + 3) + 1 && \text{by associative property} \\ &= 3(3K^2 + 4K + 1) + 1 && \text{by factoring} \end{aligned}$$

but there is an integer q such that $q = 3K^2 + 4K + 1$ by the closure property of integers for multiplication and addition, so $n^2 = 3q + 1$ by substitution. □

Explore the Model Questions

1. What is the proposition that we are attempting to prove?
2. What are the possible remainders when an integer is divided by 3?
3. How many cases must hold true in order to prove the proposition?

Critical Thinking Questions

4. Provide the proof for CASE 2: n may have a remainder of 1.

start time:

Model II. Proof by Induction (15 min)

- We are given a sequence of numbers: $1, 3, 5, 7, 9, \ldots$ and want to prove that the closed formula for the sequence is $a_n = 2n - 1$.

Explore the Model Questions

1. What would the next number in the sequence be?
2. What is the recursive formula for the sequence?
3. Is the closed formula true for a_1?

   What about a_2?

   What about a_3?

Critical Thinking Questions

4. How many values would we have to check before we could be sure that the closed formula is correct?
5. a_1 is called the base case and we can prove the base case: $a_1 = (2 \cdot 1) - 1$

   $a_1 =$ ________________ (proves the closed formula is correct for a_1)

Now if we could prove that if it is true for a_m then it is true for a_{m+1}, we could conclude that it is true for any a_n. This is the **Principle of Mathematical Induction**.

We have shown that if the closed formula is true the base case and that if it is true for the m^{th} term in the sequence, then it is true for the $(m+1)^{\text{th}}$ term.

This concept of **proof by induction** may take some time to "wrap you head around". We first showed that it was true for the base case (a_1). We then showed that if it is true for some term in the sequence, then it is true for the next term, and if it is true for the next term would then be true for the next, and the next, and the next, ...

Application

1. Prove that the sequence defined by the recursive formula: $a_k = a_{k-1} + 4$, $a_1 = 1$ and $k \geq 2$, is equivalently described by the closed formula: $a_n = 4n - 3$.

    1. Using the recursive formula, what are the first 3 terms in this sequence?

    2. Prove that the closed formula is correct for the base case, $a_1 = 1$?

    3. Prove by mathematical induction:

        $a_m = a_{m-1} + 4$ Starting out with the recursive formula expressed using m.

        You must complete the proof. You can begin by expressing the term a_{m-1} using the closed formula and substituting "$m - 1$" for "n".

STOP

Wait for all teams to complete this exercise.

start time:

Model III. Proof by Contradiction (15 min)

Proposition 2

If $n \bmod 3 = 1$, then $n \bmod 9 \neq 5$.

Proof. Suppose someday someone does find a counterexample to this statement. What will it look like? It will be an integer n such that $n \bmod 3 = 1$ and $n \bmod 9 = 5$.

This means that there will be integers K and L such that $n = 3K + 1$ and $n = 9L + 5$.

This in turn implies that

$$3K + 1 = 9L + 5$$

$$3K - 9L = 5 - 1 \qquad \text{by rearranging the terms}$$

$$3(K - 3L) = 4 \qquad \text{by factoring out a 3 and doing the arithmetic } (5 - 1 = 4)$$

$$K - 3L = \frac{4}{3} \qquad \text{dividing both sides by 3}$$

Since K and L are integers, this equation cannot be true. Therefore, a counterexample will never be found. □

Explore the Model Questions

1. In this model, what was the hypothesis?
2. What was the conclusion?
3. For a counterexample the ________________ must be true, and the ________________ false?

   [Use the terms **hypothesis** and **conclusion** to fill in the blanks.]
4. From you previous knowledge and experience, what is the meaning of $n \bmod 3 = 1$?
5. How is this expressed using K as some integer?
6. For a counterexample, what two equations must be true?

   n = ___________________________ and n = ___________________________

Critical Thinking

1. Why can't the equation $K - 3L = \frac{4}{3}$ be true?

Application

Proposition 3

For all integers a and b, $\left(\frac{a}{b}\right)^2 \neq 2$.

Proof. Suppose that someone someday finds integers a and b for which $\left(\frac{a}{b}\right)^2 = 2$.

Let's assume that the fraction $\frac{a}{b}$ has already been reduced to lowest terms, so in particular a and b are not both even.

And further assume that a is even and b is odd.

Then for some integers K and L, $a = 2K$ and $b = 2L + 1$.

$\left(\frac{a}{b}\right)^2 = 2$ can then be expressed as

$$\left(\frac{2K}{2L+1}\right)^2 = 2$$

$$\frac{4K^2}{4L^2+4L+1} = 2$$

$$4K^2 = 2 \cdot (4L^2 + 4L + 1)$$

$$2K^2 = (4L^2 + 4L + 1)$$

$$K^2 = \frac{4L^2+4L+1}{2}$$

$$K^2 = 2L^2 + 2L + \frac{1}{2}$$

$$K^2 - 2L^2 + 2L = \frac{1}{2}$$

but $K^2 - 2L^2 + 2L$ is an integer and cannot = $\frac{1}{2}$. □

Now, complete the proof for the other possibility, **a is odd and b is even**.

Looking Back - Group Reflection

Review the job descriptions on your role card. Evaluate privately on how well you performed in your role. Provide the following using the **SII framework**:

S – *Strength* (Also what specifically did you do that would indicate that it was a strength)

I – *Improvement* Area

I – *Insight* concerning either the process or the content of the activity

Facilitator

List below how long each Model took your group:

- Model I:
- Model II:
- Model III:

Process Analyst

Give feedback to each of your other team members with a **strength** that you observed, either in their role or their process skills.

1. Facilitator
2. Spokesperson
3. Qualtiy Control
4. Process Analyst

Quality Control

Summarize the activity for your group members, by answering the following questions.

1. What are the steps for a proof by mathematical induction?

2. What is the first step in a proof by contradiction?

Spokesperson

Gather up the name cards and any other materials and give them to Dr. Goadrich.