CSCI 365 Problem Set 1: Rewriting

due Friday, 26 January 2024

Specification

To receive credit for this problem set, you must complete any 7 out of the 10 exercises.

Rewriting

Exercise 1 Evaluate the below arithmetic expression via a sequence of rewriting steps. At each step, underline the subexpression to be rewritten.

$$2 - (3 - 4) \times (5 \times (3 + 2))$$

Exercise 2 Evaluate the below arithmetic expression via a sequence of rewriting steps. At each step, underline the subexpression to be rewritten.

$$9 + ((8 + ((7 + ((6 + 5) \times 4)) \times 3)) \times 2)$$

Exercise 3 Show the process of evaluating the below expression of Peano arithmetic using the rules introduced in class.

$$(\mathsf{S}(\mathsf{S}(\mathsf{SZ})) \cdot \mathsf{S}(\mathsf{SZ})) + \mathsf{S}(\mathsf{SZ})$$

As a reminder, the syntax and rewrite rules are as follows:

$$e ::= \mathsf{Z} \mid \mathsf{S}e \mid e_1 + e_2 \mid e_1 \cdot e_2$$

$$\begin{aligned} \mathsf{Z} + a &\longrightarrow a \\ (\mathsf{S}a) + b &\longrightarrow \mathsf{S}(a + b) \\ a \cdot \mathsf{Z} &\longrightarrow \mathsf{Z} \\ a \cdot (\mathsf{S}b) &\longrightarrow a + (a \cdot b) \end{aligned}$$

Exercise 4 Let's add an extra symbol to the syntax of Peano arithmetic from the previous exercise:

$$e ::= \mathsf{Z} \mid \mathsf{S}e \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e_1 \uparrow e_2$$

Suppose we want $\uparrow$ to denote exponentiation, that is, $a \uparrow b$ should denote a^b. For example, we should have

$A \longrightarrow^* B$ means "A can be rewritten via zero or more steps to eventually reach B".

$$\mathsf{S(S(SZ))} \uparrow \mathsf{S(SZ)} \longrightarrow^* \mathsf{S(S(S(S(S(S(S(S(SZ))))))))}$$

since $3^2 = 9$. Devise appropriate rewrite rule(s) which encode the desired computational behavior.

Exercise 5 Consider the following rewriting system. First, the syntax of expressions is given by

$$c ::= \mathsf{S} \mid \mathsf{K} \mid \mathsf{I} \mid c_1 c_2$$

That is, an expression c is either S, K, I, or two expressions next to each other. As usual we also allow parentheses for disambiguation. For example, valid expressions include KI, (IS)K, I(SK), and I((KS)I).

Now consider the following rewrite rules, where s, t, and u represent arbitrary expressions:

$$\begin{aligned} \mathsf{I}s &\longrightarrow s \\ (\mathsf{K}s)t &\longrightarrow s \\ ((\mathsf{S}s)t)u &\longrightarrow (su)(tu) \end{aligned}$$

Evaluate each of the following expressions via a series of rewrites.

(a) ((SK)K)I

(b) ((SK)I)((KI)S)

(c) ((SI)I)K

Exercise 6 Design a syntax and rewrite system for expanding and simplifying polynomial expressions. For example, your rewrite system should be able to handle expressions like $2x + 3x^2 - 5x$ or $(y+3)(y-7)xy$, and it should simplfy the first example to $3x^2 - 3x$, and expand the second example to $xy^3 - 4xy^2 - 21xy$.

If it's easier, you may simply use repeated multiplication instead of exponents, for example, you could represent y^3 as yyy.

Substitution

Exercise 7 Consider the following grammar:

$$T ::= \alpha \mid \mathsf{Int} \mid \mathsf{Bool} \mid T + T \mid T \times T \mid T \to T$$

In the above grammar, α stands for any variable.

(a) Write down three example expressions of this grammar. Taken together, your examples should cover all 6 alternatives in the grammar (that is, at least one example should contain Int, at least one example should contain $\to$, and so on).

(b) Evaluate: $[b \mapsto \mathsf{Int}](b \to (\mathsf{Int} \to b))$

Be careful with parentheses!

(c) Evaluate: $[\alpha \mapsto (\mathsf{Int} \to \mathsf{Int})](\alpha \times (\mathsf{Bool} \to \alpha))$

(d) Evaluate: $[\gamma \mapsto (\mathsf{Bool} \times \mathsf{Int})]([\beta \mapsto (\mathsf{Int} + \gamma)](\beta + \mathsf{Bool} + \beta))$

Exercise 8 Evaluate each of the following expressions via a sequence of rewriting steps.

(a) $\mathsf{let}\ x = 2\ \mathsf{in}\ x + 3$

(b) $\mathsf{let}\ y = 2 + 5\ \mathsf{in}\ (\mathsf{let}\ z = y + 3\ \mathsf{in}\ z + (\mathsf{let}\ e = 1\ \mathsf{in}\ e + e) \cdot (z + y))$

(c) $\mathsf{let}\ z = 174 \cdot 93\ \mathsf{in}\ \mathsf{let}\ m = 7942 \cdot 7841\ \mathsf{in}\ 6$

(d) $\mathsf{let}\ a = (\mathsf{let}\ b = (\mathsf{let}\ c = 1\ \mathsf{in}\ c + 2)\ \mathsf{in}\ b + 5)\ \mathsf{in}\ a + 1$

Further Exploration

Exercise 9 A *confluent* rewriting system, informally, is one in which from any starting expression, any valid sequence of rewrites will always ultimately yield the same result. For example, $(1+2) \times (3+4)$ will always ultimately yield 21, regardless of whether we choose to rewrite it first to $3 \times (3+4)$ or to $(1+2) \times 7$. All of the rewriting systems we have considered as examples so far are confluent, although in general, confluence is a very special property.

The formal definition is a bit more technical, but the intuition is all we need at this point.

Make up an example of a rewriting system that is *not* confluent, and explain/demonstrate why.

Exercise 10 Demonstrate some different options for how we might evaluate the following expression via rewriting:

$$\mathsf{let}\ x = 5\ \mathsf{in}\ (\mathsf{let}\ x = 9\ \mathsf{in}\ x + 1)$$

What is difficult about this expression? What do you think the result "should" be? How might we constrain or refine our rewriting rules so that we always get the desired result in this situation?