Lambda Calculus

1935? Alonzo Church (based on Fregre, Schönfinkel ...)

goal: foundation for mathematics

failed? ⟶ foundation for computation.

both theory + practice!

Syntax

any variable

$$t ::= x \qquad \text{variable}$$
$$\mid \lambda x.\, t \qquad \text{function taking } x \text{ as input, returning } t$$
$$\mid t_1\, t_2 \qquad \text{function application}$$

eg.

y

$f\ x$

$(\lambda y.\, y\ x)$

$f\,(((\lambda z.\, z)\ u)\ (\lambda w.\, (\lambda m.\, w)))$

Conventions:

- body of λ extends as far right as possible

  eg. $\lambda y.\, y\ x$ means $\lambda y.\,(y\ x)$, NOT $(\lambda y.\, y)\ x$

- we can abbreviate multiple nested lambdas: (Currying)

  $\lambda x.\,(\lambda y.\,(\lambda z.\, t)) = \lambda x y z.\, t$

- Application associates to the left.

  $f\ a\ b\ c = ((f\ a)\ b)\ c$

Rewrite rules

$$(\lambda x.\, t_1)\ t_2 \longrightarrow [x \mapsto t_2]\, t_1$$ (*)

lambda applied to argument $\longrightarrow$ body t_1, but with x replaced by t_2.

eg. $(\lambda z.\, f\ z)\ (x\ y)$

$$\longrightarrow [z \mapsto x\ y](f\ z) = f\ (x\ y)$$

$(\lambda g.\, g\ r)\ (\lambda u.\, u)$

$$\longrightarrow [g \mapsto (\lambda u.u)](g\ r) = (\lambda u.\, u)\ r$$

$$\longrightarrow [u \mapsto r]\, u = r$$

$$(\lambda z.\, q)\ (x\ y) \longrightarrow [z \mapsto x\ y]\, q = q.$$